PHYS 393.    January 2006    Solutions.

1a.

$5\epsilon$
$4\epsilon$
$3\epsilon$
$2\epsilon$
$\epsilon$
$0$

$N = 4$ particles

$U = 5\epsilon$ energy

| | Distribution | $0$ | $\epsilon$ | $2\epsilon$ | $3\epsilon$ | $4\epsilon$ | $5\epsilon$ | (ii) No microstates ↓ |
|---|---|---|---|---|---|---|---|---|
| (i) | 1 | 3 | 0 | 0 | 0 | 0 | 1 | 4 |
| | 2 | 2 | 1 | 0 | 0 | 1 | 0 | 12 |
| | 3 | 2 | 0 | 1 | 1 | 0 | 0 | 12 |
| | 4 | 1 | 2 | 0 | 1 | 0 | 0 | 12 |
| | 5 | 1 | 1 | 2 | 0 | 0 | 0 | 12 |
| | 6 | 0 | 3 | 1 | 0 | 0 | 0 | 4 |
| | | | | 2 | | | | Total = 56 |
| | | | | | | | | 2 |

Number of microstates in total = 56.

Mean occupation

(iii) $n_0 = \frac{1}{56}\left\{4\times 3 + 12\times 2 + 12\times 2 + 12\times 1 + 12\times 1 + 4\times 0\right\} = \frac{84}{56} = 1.50$

$n_1 = \frac{1}{56}\left\{4\times 0 + 12\times 1 + 12\times 0 + 12\times 2 + 12\times 1 + 4\times 3\right\} = \frac{60}{56} = 1.07$

$n_2 = \frac{1}{56}\left\{4\times 0 + 12\times 0 + 12\times 1 + 12\times 0 + 12\times 2 + 4\right\} = \frac{40}{56} = 0.71$

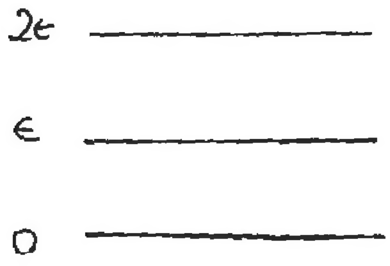
$n_3 = \frac{1}{56}\left\{4\times 0 + 12\times 0 + 12\times 1 + 12\times 1 \ldots\right\} = \frac{24}{56} = 0.43$

2

$n_4 = \frac{1}{56}\left|12\right| = \quad = 0.21$

$n_5 = \frac{1}{56}\times 4 = \quad = 0.07$ [6]

1 (b). N atoms at temp T in level scheme shown




(i) Partition function $Z = \sum_j \exp(-\epsilon_j/kT)$

$= 1 + \exp(-\epsilon/kT) + \exp(-2\epsilon/kT)$ 2

(ii) $U = NkT^2 \frac{\partial}{\partial T} \left\{ \ln\left[1 + \exp(-\epsilon/kT) + \exp(-2\epsilon/kT)\right]\right\}$

$$= NkT^2 . \frac{\left\{ (\epsilon/kT^2) \exp(-\epsilon/kT) + (\frac{2\epsilon}{kT^2}) \exp(-2\epsilon/kT)\right\}}{\left[1 + \exp(-\epsilon/kT) + \exp(-2\epsilon/kT)\right]}$$

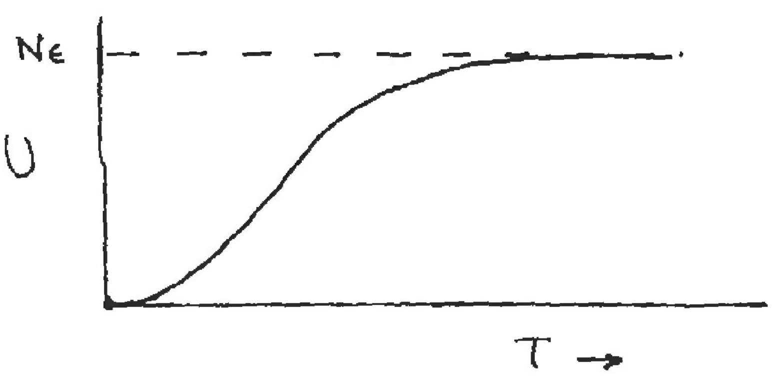
$$= NkT^2 . \frac{(\epsilon/kT^2) \exp(-\epsilon/kT)\left[1 + 2\exp(-\epsilon/kT)\right]}{\left[1 + \exp(-\epsilon/kT) + \exp(-2\epsilon/kT)\right]}$$

$$= \frac{N\epsilon . \exp(-\epsilon/kT)\left[1 + 2\exp(-\epsilon/kT)\right]}{\left[1 + \exp(-\epsilon/kT) + \exp(-2\epsilon/kT)\right]}$$ 2

(iii) As $T \to 0$ $\exp(-\epsilon/kT) \to 0$ $U \to 0$

As $T \to \infty$ $\exp(-\epsilon/kT) \to 1$ $U \to N\epsilon . \frac{1}{3} . 3 \to N\epsilon.$ 2

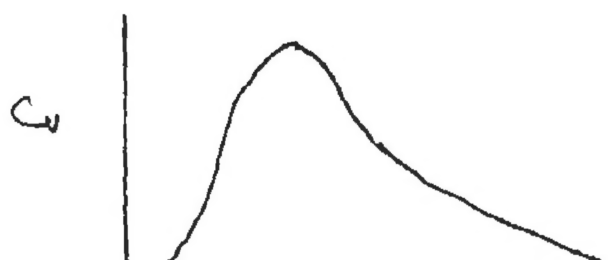
(iv)




Sketch from above values.

2

(V)



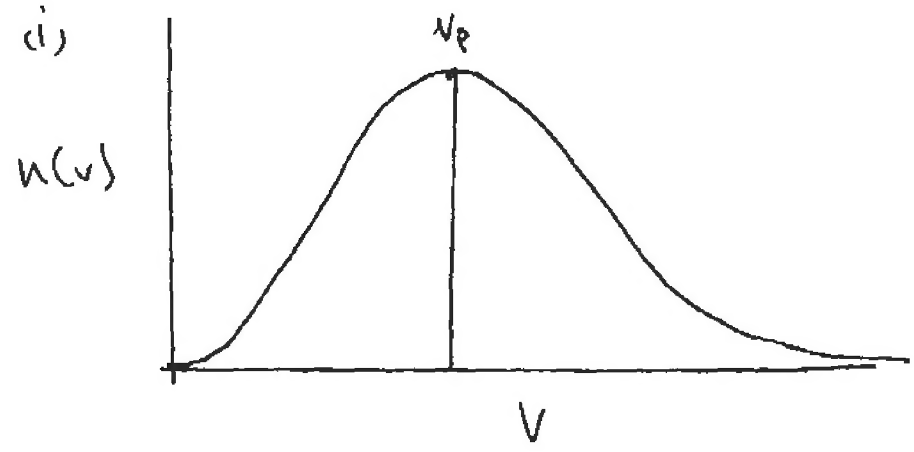

Sketch from slope of U vs T curve

2

[10]

1c. $$n(v) = 4\pi N\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right).$$

(i)




2

(ii) Most probable velocity $v_p$

$$\frac{dn}{dv} = 0 = 4\pi N\left(\frac{m}{2\pi kT}\right)^{3/2}\left\{v^2 \cdot -\frac{2mv}{2kT}\exp\left(-\frac{mv^2}{2kT}\right) + 2v\exp\left(-\frac{mv^2}{2kT}\right)\right\}$$

$$0 = -\frac{\cancel{2}mv^{\cancel{3}2}}{\cancel{2}kT} + 2\cancel{v}$$

$$v_p^2 = \frac{2kT}{m} \qquad v_p = \left(\frac{2kT}{m}\right)^{1/2}$$ 2

(iii) $$\overline{v_m^2} = \frac{\int_0^\infty \cancel{4\pi N\left(\frac{m}{2\pi kT}\right)^{3/2}} \cdot v^4 \exp\left(-\frac{mv^2}{2kT}\right)dv}{\int_0^\infty \cancel{4\pi N\left(\frac{m}{2\pi kT}\right)^{3/2}} v^2 \exp\left(-\frac{mv^2}{2kT}\right)dv} = \frac{I_4}{I_2}$$

$b = m/2kT.$

$$= \frac{3}{2b}\cdot\frac{\cancel{I_2}}{\cancel{I_2}} = \frac{3}{2m}\cdot 2kT = \frac{3kT}{m}.$$ 2

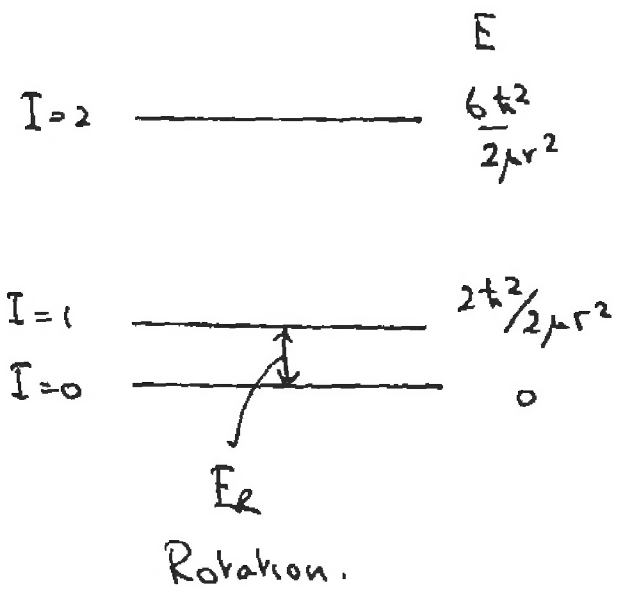
(iv) $$U_{mole} = \frac{1}{2}m\overline{v_m^2}\cdot N_A = \frac{\cancel{m}}{2}\cdot\frac{3kT}{\cancel{m}}\cdot N_A = 1.5\times1.38\times10^{-23}\times300\times6\times10^{23}$$

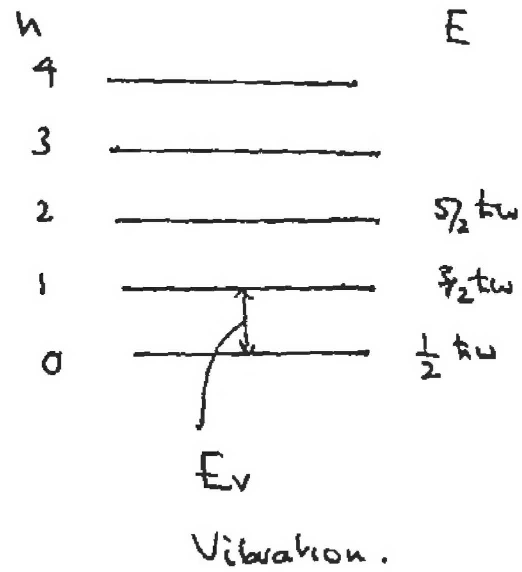
$$= 3726\ \text{J}.$$

2

[8]

1.d Diatomic molecules. HD.




(i) $E_R = \frac{\cancel{2}\hbar^2}{\cancel{2}\mu r^2}$ . Reduced mass $\mu = \frac{m \cdot 2m}{m+2m} = \frac{2}{3}m$

$= 0.67 \times 1.66 \times 10^{-27}$ kg.

$$E_R = \left(\frac{6.63\times10^{-34}}{2\pi}\right)^2 \cdot \frac{1}{0.67\times1.66\times10^{-27}\times(1.05\times10^{-10})^2} = 9.08\times10^{-22}\,\text{J}\ (5.68\times10^{-3}\,\text{eV})$$

$$E_V = \hbar\left(\frac{B}{\mu}\right)^{1/2} = \frac{6.63\times10^{-34}}{2\pi}\left(\frac{6.91\times10^{2}}{0.67\times1.66\times10^{-27}}\right)^{1/2} = 8.32\times10^{-20}\,\text{J}\ (0.52\,\text{eV})$$

2

(ii) Rotational motion excited at $\Theta_R$

where $k\Theta_R = 9.08\times10^{-22}$ $\Theta_R = \frac{9.08\times10^{-22}}{1.38\times10^{-23}} = 65.8$ K

Vibrational motion at

$k\Theta_V = 8.32\times10^{-20}$ J $\Theta_V = \frac{8.32\times10^{-20}}{1.38\times10^{-23}} = 6028$ K.

2

(iii) At $T = 20$K.

Translational motion contributes $3R/2$ to molar $C_V$

Rot$^{le}$ and vibrational motions not excited thus $C_V \sim 3R/2$ 2

At $T = 300$K

Translational and rotational motions excited $C_V = \underbrace{\frac{3R}{2} + R}_{\text{trans + rot}^{le}} \simeq 5R/2$

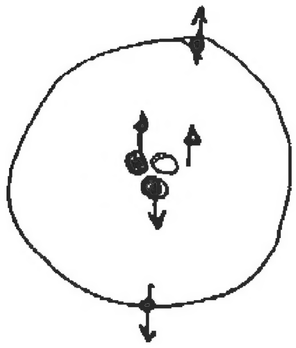
Vibrational motion not excited. 2

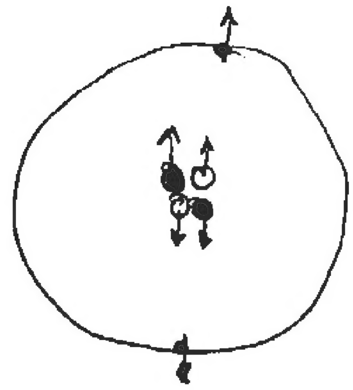
[8]

1a.

Constituents $He^3$ atom $He^4$ atom

(i)





2 (1s) electrons spins ↑↓
2 protons spins ↑↓
1 neutron spin ↑

2 (1s) electrons spins ↑↓
2 protons spins ↑↓
2 neutrons spins ↑↓

2

(ii) Electronic ang mtm $J=0$
Nuclear ang mtm $I=\frac{1}{2}$
Total $F = \underline{I} + \underline{J} = \frac{1}{2}$

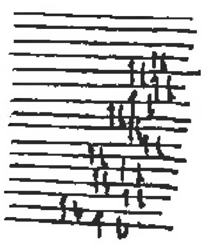
Electronic ang mtm $J=0$
Nuclear ang mtm $I=0$
Total $\underline{F} = \underline{I} + \underline{J} = 0$.

2

(iii) Atoms with half integer ang mtm – like $He^3$ are fermions
Atoms with 0 or integer ang mtm (like $He^4$) are bosons.

2

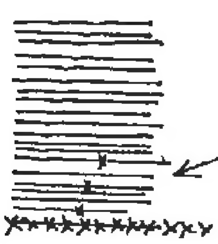
(iv) $He^3$ atom populations at low temp. $(T = 2K)$



← up to Fermi energy $\epsilon_F$.
← each state has ↑↓ $He^3$ atoms only

2

(v) $He^4$ atom populations at low temp $(T = 2K)$



Small number $He^4$ in excited states

Very many condensed into ground state.

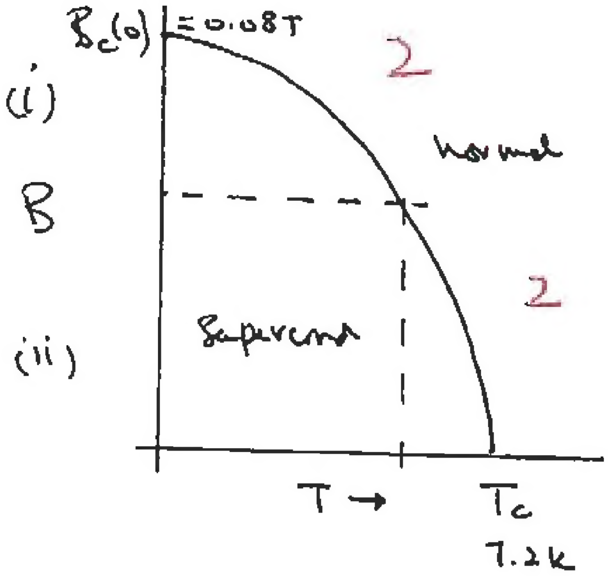
2

[10]

1f.

(i)

(ii)




2

2

(iii) $B_c(T) = B_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$

$0.05 = 0.080\left[1 - \left(\frac{T}{7.2}\right)^2\right]$

$0.625 = 1 - \left(\frac{T}{7.2}\right)^2$

$\left(\frac{T}{T_c}\right)^2 = 1 - 0.625 = 0.375$

$\frac{T}{T_c} = 0.612$

$T = 0.612\,T_c = 0.612 \times 7.2 = 4.41\text{ K}.$ 2

(iv) At $T = 5.6$ K — calculate critical field

$B_c(T) = 0.080\left[1 - \left(\frac{5.6}{7.2}\right)^2\right]$

$= 0.080\left[1 - 0.605\right] = 0.080 \times 0.395$

$= 0.0316$ T. 2

Since $B = 0.035$ T is greater than this

then at $B = 0.035$ T and $T = 5.6$ K — Pb is normal

[8]

1g.

$$E = \pm \mu B = \pm\ 1.0 \times 9.27 \times 10^{-24} \times 1.5$$

$$= 1.391 \times 10^{-23}\ \mathrm{J}$$

(i) Splitting $= 2\mu B = 2.78 \times 10^{-23}\ \mathrm{J}.$

(ii) Population ratio $\dfrac{n_1}{n_0} = \exp\left(-\dfrac{2\mu B}{kT}\right)$ $\qquad T = 4.2\mathrm{K}.$

$$= \exp\left(-\frac{2.78 \times \cancel{10^{-23}}}{1.38 \times \cancel{10^{-23}} \times 4.2}\right)$$

$$= \exp(-0.480)$$

$$= 0.62.$$

$0.62n_0$ —— $1.0 \times 9.27 \times 10^{-24} \times 10^{-4}\ \mathrm{T}$

$n_0$ —— $0$

$$0.62 = \exp\left(-\frac{9.27 \times 10^{-28}}{kT}\right)$$

$$\ln(0.62) = -\frac{9.27 \times \cancel{10^{-24}} \times 10^{-5}}{1.38 \times \cancel{10^{-23}} \times T}$$

$$-0.48 = -\frac{9.27 \times 10^{-5}}{1.38 \times T} \qquad T = \frac{9.27 \times 10^{-5}}{0.48 \times 1.38} = 1.4 \times 10^{-4}\ \mathrm{K}.$$

2(a.)

(i) For motion in a box of side L

Using periodic boundary conditions. $\exp(ik_xL) = 1.$

Get quantised values $k_x = \frac{2\pi}{L} n_x$

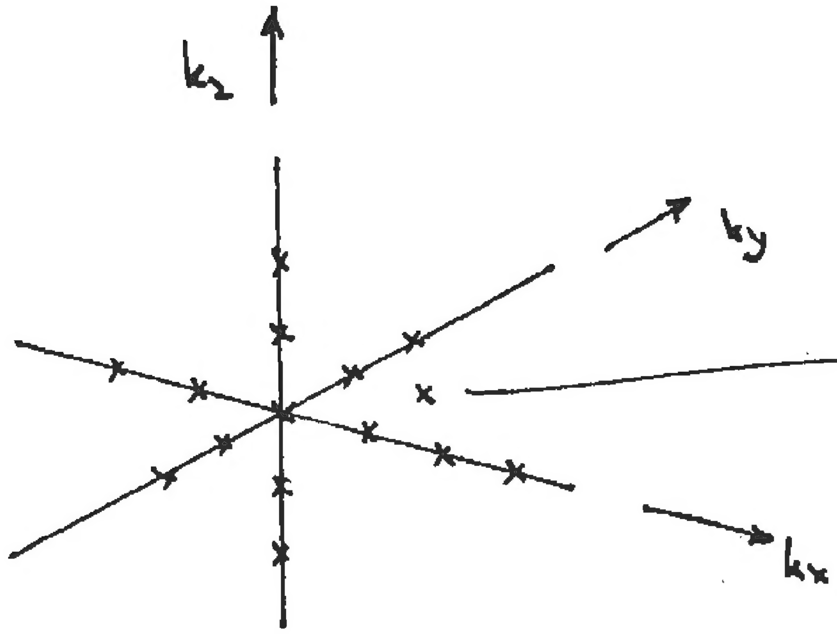
$k_y = \frac{2\pi}{L} n_y$

$k_z = \frac{2\pi}{L} n_z$

where $n_x, n_y, n_z$ are $0, \pm 1, \pm 2$ — integers.

2

(ii)




point represents quantised motion specified by q/n $n_x, n_y, n_z$.

2

(iii) Number of states $g(k)\,dk$ lying between $k \rightarrow k+dk$

$$g(k)\,dk = \frac{\text{volume in } k \text{ space}}{\left(\frac{2\pi}{L}\right)^3}$$ (1)

volume / mode (1)

Volume in k space is spherical shell — radius k and thickness dk — volume $= 4\pi k^2\,dk.$ (1)

Thus for spinless particles

$$g(k)\,dk = \frac{4\pi k^2\,dk}{\left(\frac{2\pi}{L}\right)^3}$$

For electrons with ↑↓ occupation of each point (1)

$$g(k)\,dk = \frac{2\,V.\,4\pi k^2\,dk}{(2\pi)^3} \quad \text{where } V = L^3.$$

4

(iv) For free electron $\epsilon = \frac{\hbar^2 k^2}{2m}$ . 1

(v) $k^2 = \frac{2m\epsilon}{\hbar^2}$ thus $\cancel{2} k\,dk = \frac{\cancel{2}m}{\hbar^2}.d\epsilon$ (1)

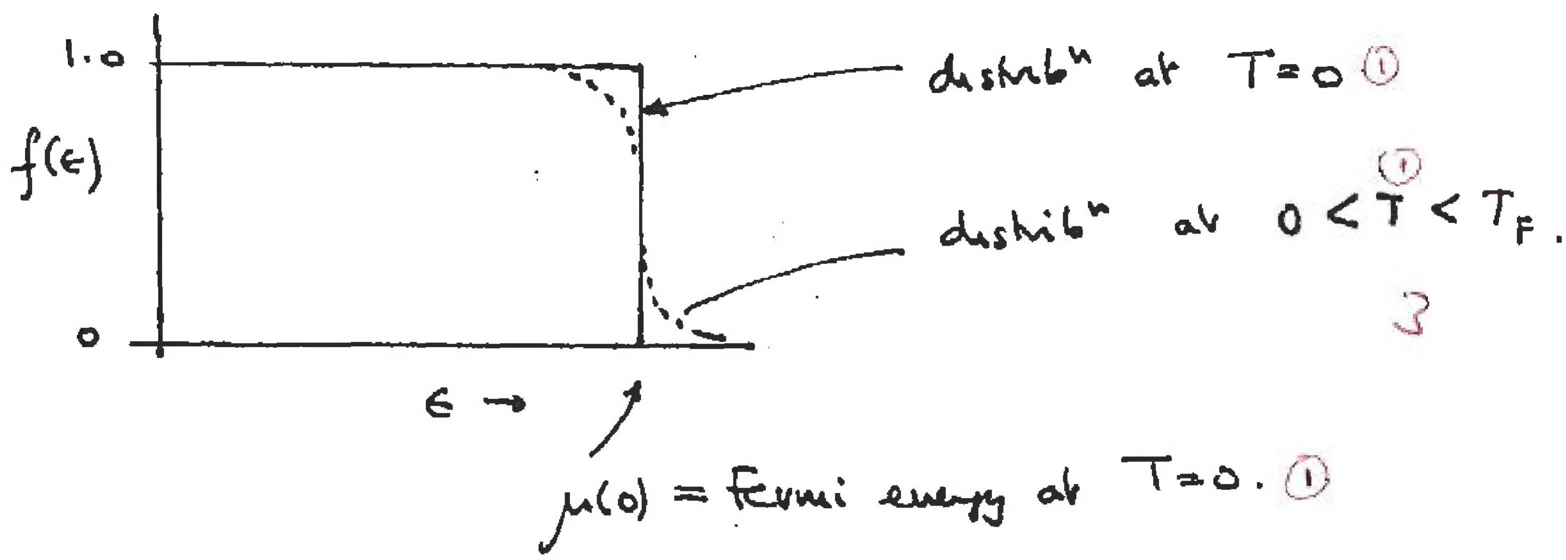
$$k = \left(\frac{2m\epsilon}{\hbar^2}\right)^{1/2} \quad (1)$$

Thus $g(k)\,dk = 2.\frac{V.\ 4\pi k^2}{(2\pi)^3}.dk$

transforms to $g(\epsilon)\,d\epsilon = 2.V.\frac{4\pi}{(2\pi)^3}.\left(\frac{2m\epsilon}{\hbar^2}\right)^{1/2}.\frac{m\,d\epsilon}{\hbar^2}$

$$g(\epsilon)\,d\epsilon = 2V.\frac{1}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}\,d\epsilon \quad (1) \quad 3$$

(vi)




3

(vii) At $T=0$ all states with ~~energy~~ $k < K_F$ are filled with $\uparrow\downarrow$ electrons

$$N = \underbrace{\frac{4}{3}\pi k_F^3}_{\text{vol k space}} . \underbrace{2}_{\uparrow\downarrow} . \underbrace{\frac{V}{(2\pi)^3}}_{1/\text{vol / state}} \quad (1)\ (1)$$

$$\therefore\ k_F^3 = (2\pi)^3.\frac{3N}{4\pi.2V} = \frac{3\pi^2 N}{V} \quad (1)$$

Fermi energy $\mu(0) = \frac{\hbar^2 k_F^2}{2m} = \left(\frac{\hbar^2}{2m}\right).\left(\frac{3\pi^2 N}{V}\right)^{2/3}$ (1) 4

(viii) Silver — molar volume $= 10.27 \times 10^{-6}\ m^3$

Thus $\frac{N}{V} = \frac{6 \times 10^{23}}{10.27 \times 10^{-6}}$ (1)

$$\mu(0) = \left(\frac{\hbar^2}{2m}\right)\left(\frac{3\pi^2 \times 6 \times 10^{23}}{10.27 \times 10^{-6}}\right)^{2/3}$$

$$= \left(\frac{6.63 \times 10^{-34}}{2\pi}\right)^2 \cdot \frac{1}{2 \times 9.11 \times 10^{-31}} \left(\frac{3\pi^2 \times 6 \times 10^{23}}{10.27 \times 10^{-6}}\right)^{2/3} \quad (1)$$

$$= 9.01 \times 10^{-19}\ J \quad (1) \quad (5.63\ eV).$$

$$\equiv 5.63\ eV .$$

3

(ix) $C_V$ at 5K.

$$C_V = \left(\frac{\partial U}{\partial T}\right) \overset{(1)}{} = \frac{2N\pi^2 k^2 T}{4\mu(0)}. \quad (1)$$

$$= \frac{2 \times 6 \times 10^{23}\pi^2 (1.38 \times 10^{-23})^2 \times 5}{4 \times 9.01 \times 10^{-19}} =$$

$$= 3.13 \times 10^{-3}\ J K^{-1}. \quad (1)$$

3

2b.

(i) Radiation in equilibrium within a cavity at temperature T Quantised assumes a frequency distribution $g(\nu)$ determined only by T. This is black body radiation. 4

(ii) Relate k and $\nu$

$$k = \frac{2\pi}{\lambda} \quad , \quad \nu = \frac{c}{\lambda} \qquad \text{thus} \quad k = \frac{2\pi\nu}{c}$$

$$dk = \frac{2\pi}{c} \cdot d\nu$$

Thus from $g(k)\,dk = 2 \cdot V \cdot \frac{4\pi k^2 dk}{(2\pi)^3}$

Get
$$g(\nu)\,d\nu = 2 \cdot V \cdot \frac{4\pi}{(2\pi)^3} \cdot \frac{4\pi^2\nu^2}{c^2} \cdot \frac{2\pi}{c}\,d\nu$$

$$= \frac{8V\pi\nu^2\,d\nu}{c^3}$$

3

(iii) $h\nu$ is photon energy 1

$[\exp(h\nu/kT) - 1]^{-1}$ is probability of state of energy $h\nu$ is occupied - Boson occupation factor!

(iv) As $\nu \to 0$
$$[\exp(h\nu/kT) - 1]^{-1} \to \frac{1}{[\cancel{1} + h\nu/kT - \cancel{1}]} \to \frac{kT}{h\nu}$$

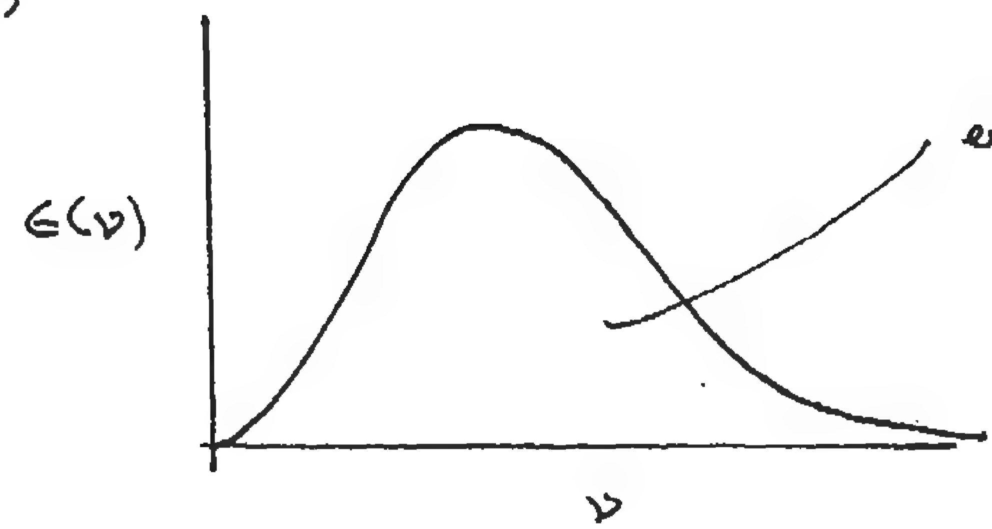
thus $\epsilon(\nu) \propto \nu^2 \to 0$ as $\nu \to 0$. 2

As $\nu \to \infty$
$\exp(h\nu/kT) \to \infty$ $\quad \epsilon(\nu) \to 0$. 2

$\exp(h\nu/kT)$ in denominator outweighs $\nu^2$ in numerator

(2) for curve

(v)




energy density (2) $\frac{U}{V}$ is integrated area under curve.

4

(vi) $$\frac{U}{V} = \int_0^\infty \frac{8\pi \text{[redacted]} h\nu^3 \, d\nu}{[\exp(h\nu/kT)-1]c^3}$$

put $y = \frac{h\nu}{kT}$ (1)

$dy = \left(\frac{h}{kT}\right) d\nu$ (1)

$\nu = y\frac{kT}{h}$

$$\frac{U}{V} = \int_0^\infty \frac{8\pi \text{[redacted]} h \cdot y^3}{c^3[\exp(y)-1]} \cdot \left(\frac{kT}{h}\right)^3 \cdot \left(\frac{kT}{h}\right) dy$$

$$= \frac{8\pi \text{[redacted]} h}{c^3}\left(\frac{kT}{h}\right)^4 \cdot \int_0^\infty \frac{y^3\,dy}{[\exp(y)-1]}$$ (1)

$$= 8\pi \text{[redacted]} \frac{(kT)^4}{(hc)^3} \cdot \frac{\pi^4}{15} = \frac{8\pi^5}{15}\frac{(kT)^4}{(hc)^3}.$$ (1) 4

(vii) At $T = 1000\,K$

$$\frac{U}{V} = \frac{8\pi^5 (1.38\times10^{-23}\times10^3)^4}{15\ (6.63\times10^{-34}\times3\times10^8)^3} = 7.60\times10^{-4}\ J\,m^{-3}.$$ 2

(viii) $\left(\frac{U}{V}\right) \propto T^4$

$$\frac{\left(\frac{U}{V}\right)_T}{\left(\frac{U}{V}\right)_{1000}} = 10 = \frac{T^4}{(1000)^4} \qquad T^4 = 10^{13}$$

$T = 1778\ K.$ 2

3a. Superconductivity.

(i) Basic features — due to boson condensation of current carriers (1)

— bosons are electron pairs (1)

— energy gap between ground + excited states. (1)

— so collisions do not excite pairs — so no resistance. (1)

— pairs (Cooper pairs) have $L = S = 0$. (1) 4

(ii) Current carriers in normal phase are single electrons (1)

In superconducting phase are electron pairs with $L=0$ $S=0$ $\uparrow\downarrow$. (1) 2

(iii) Critical temperature occurs when $kT$ breaks pair bonding (1)

Critical field occurs when applied field action on electrons in pair breaks pair bonding. (1)

Once pair bonds broken — individual electrons are fermions — cannot condense ~~ent~~ superconducting state. (1) 3

(iv) Isotope effect.

Seen exptly that $T_c \propto \frac{1}{\sqrt{m}}$ (1)

Arises from lattice excitation where $\omega_j = \sqrt{\frac{K}{m}}$. (1)

Shows that lattice plays role in pairing of electrons to form Cooper pairs (1) 3

(v) Cu has smaller resistivity than normal phase of superconductor Pb. (1)

This because interaction of single electron with lattice is smaller in Cu — good for normal conductivity. (1)

In Pb — larger interaction lattice - electron causes more scattering — higher resistivity in normal phase but also pairing to give rise to superconductivity (1) 3

(vi) Nature of high $T_c$

Ceramics – planes of Cu – O (1) spaced apart by other rows

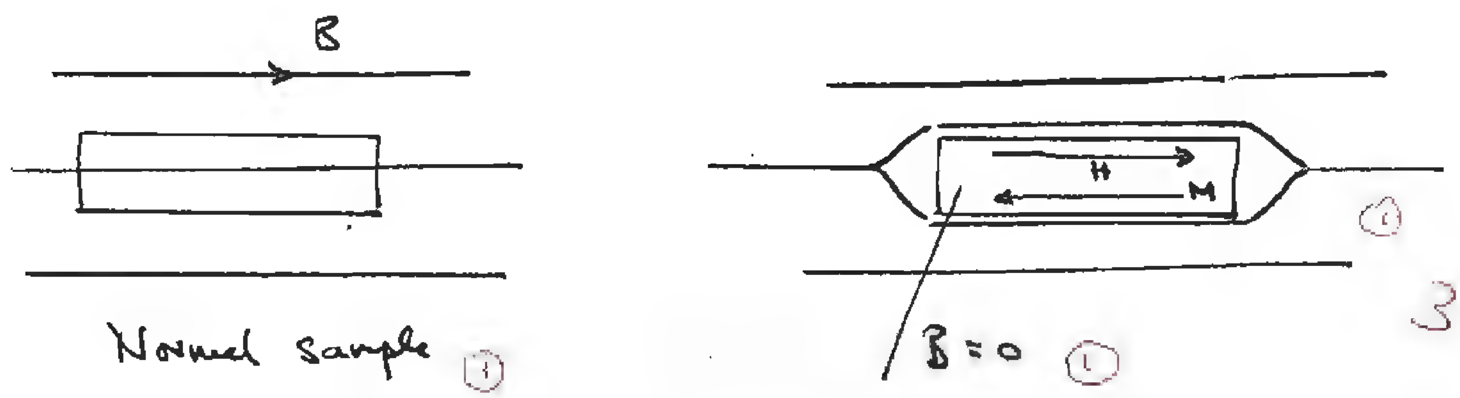
eg $YBa_2Cu_3O_{7-x}$. (1)

Cu is in mixed valence state – allows hole-pairs to form and move on Cu – O planes. (1)

Pairing mechanism not yet fully understood. 3

(vii) Meissner effect.

For sample in external magnetic field – when sample goes Superconducting – all flux of B expelled




3

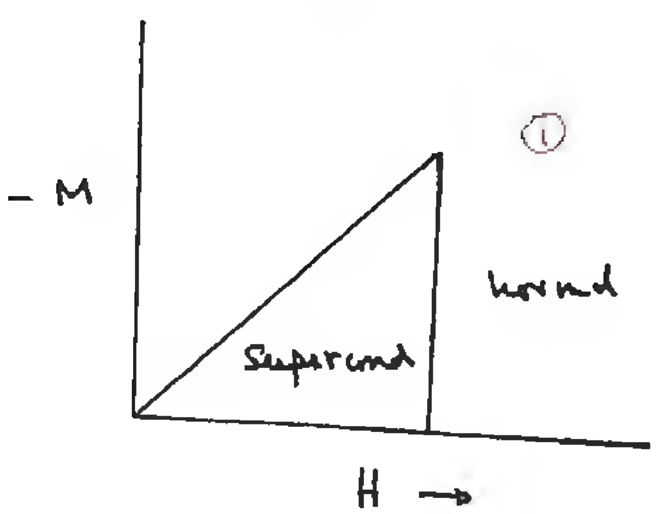
In sample $B = 0$

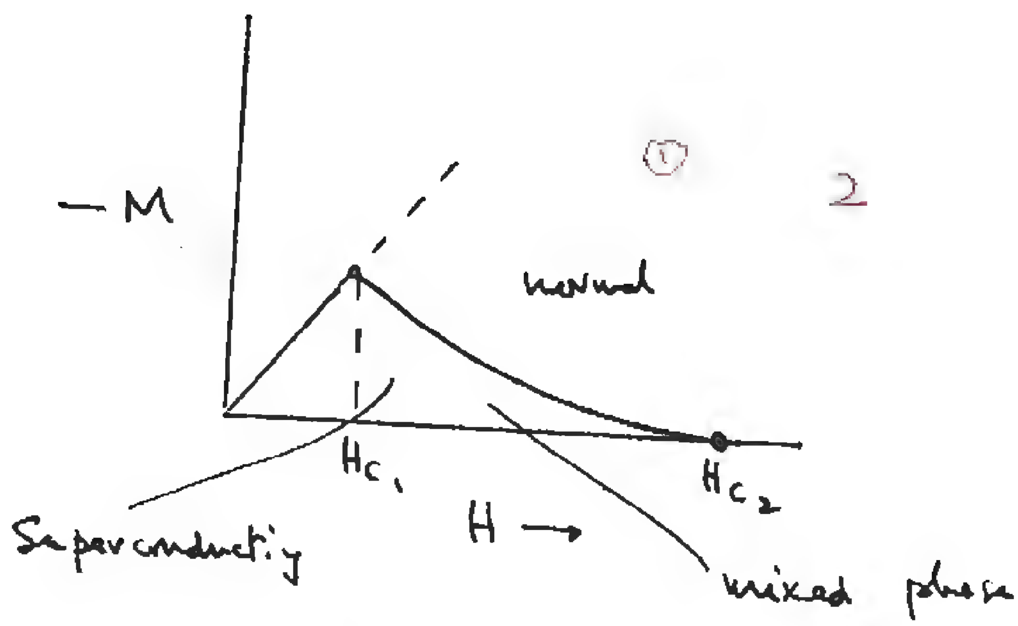
Since $M = -H$

and $B = \mu_0(H + M)$.

(viii)

Type I



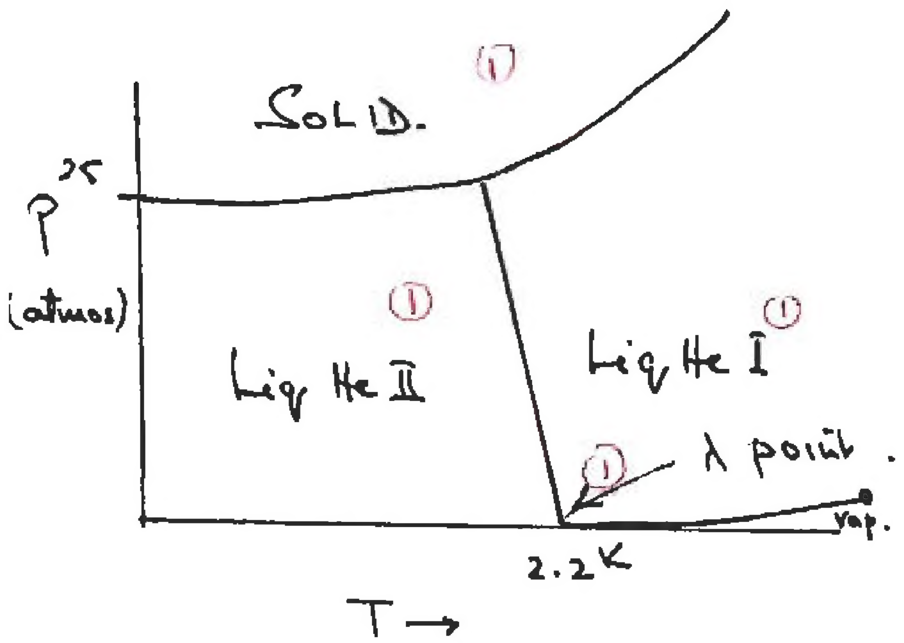

2

(ix) Examples. Type I

– elements Nb, Pb, Al (1)

Type II alloys $Nb_3Sn$ or high $T_c$ $YBa_2Cu_3O_{7-x}$. (1) 2

3b. $He^4$




4

Liquid He II is considered by 2 fluid model

2 fluid — normal
— Superfluid

changes proportions — all superfluid at 0K
all normal at λ point (2.2K

Superfluid ($He^4$ condensate) carries no entropy
has no viscosity.

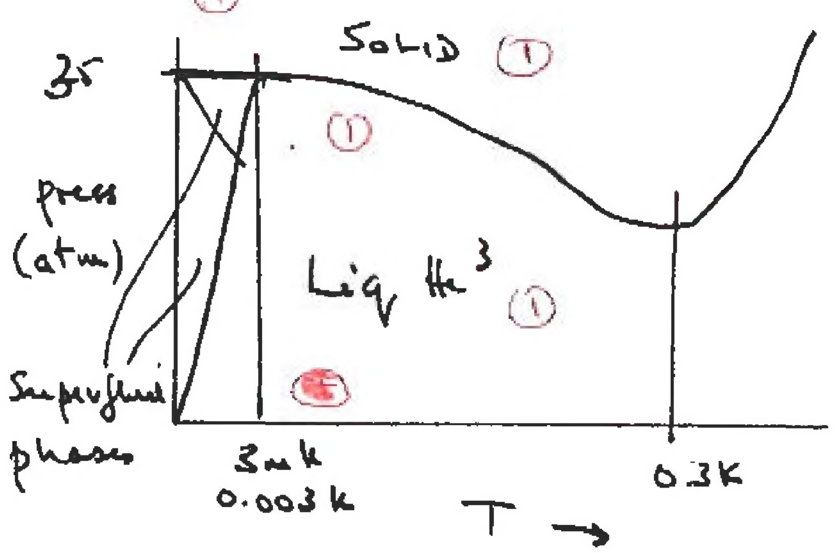
Viscosity measurements.

Torsionally oscillating discs in Liq He II – measured normal component attached to discs. – get large value for η

Flow thru narrow capillary – superfluid flows without viscosity
– get only ~~upper~~ limit for η.

2 fluid model explains as measuring diff components of Liq He II.

4




$He^3$

4

Liq $He^3$ shows superfluid properties at $T < 3mK$. ①
This occurs because 2 $He^3$ atoms (fermions) can pair ~~to~~ form bosons that can then condense to ground state and become a superfluid phase. ①
Mechanism of pairing not well understood.

Phase diagram (superfluid part) is sensitive to magnetic field because pairs have $S=1$ and ① $L=1$. — they are magnetic. 2
Different superfluid phases are differently magnetic and in applied field the phase that can ① interact with field to get to lowest energy increases — other phase decreases.
2

Low temperature.

— either Pomeranchuk cooling

or $He^3/He^4$ dilution refridgerator

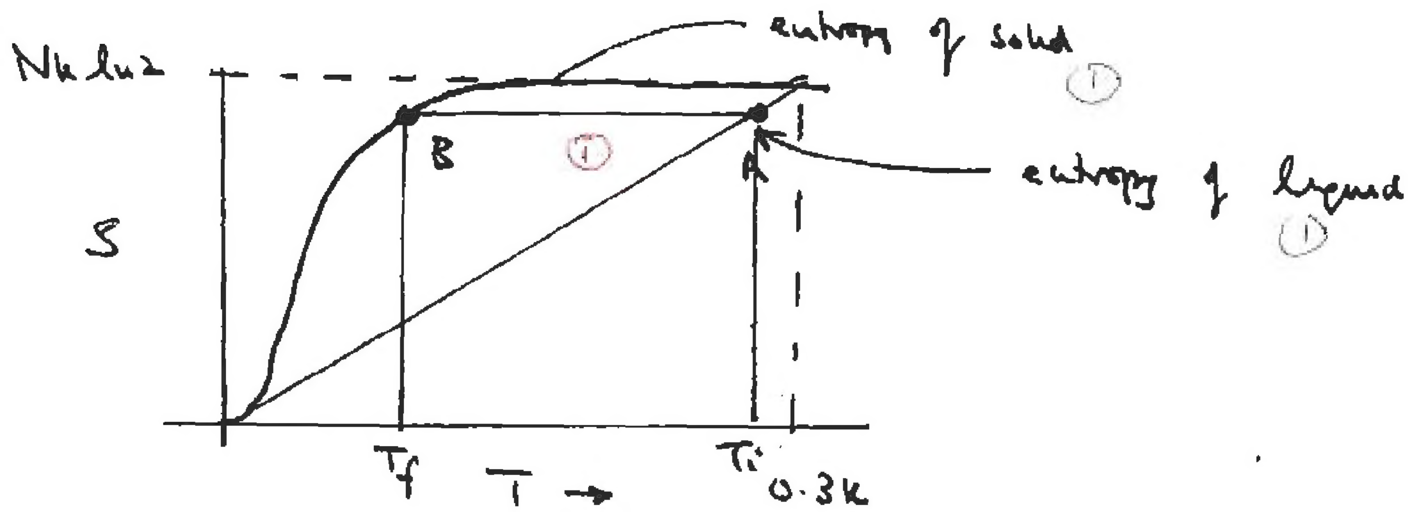
EITHER

Pomeranchuk cooling.

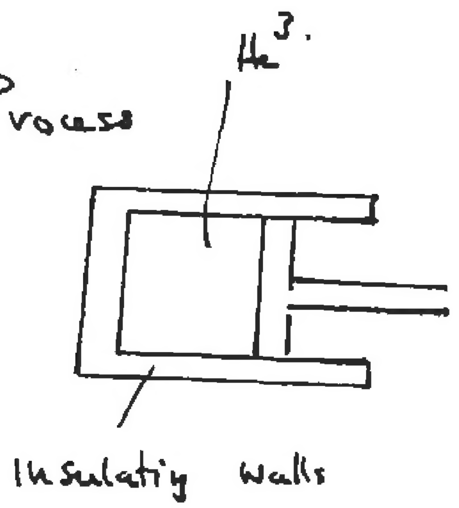
Uses liquid / solid $He^3$.

Theory. In temp range $0 \to 0.3K$ entropy $S$ of solid $He^3$ is greater than liquid $He^3$.

This because main entropy is in nuclear spins — this greater in Boltzmann distribn in distinguishable atoms than in Fermi-Dirac distribn in indistinguishable liq atoms.




Process




– compress liq $He^3 \longrightarrow$ solid $He^3$ adiabatically (const S).

– this reduces temp $T_i \to T_f$.

In practice start ~ ~~...~~ 0.2K and will end at ~10 mK.

One shot process. – connect sample to cell via conducting path

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OR

Helium dilution refrigerator

(i) Form of helium – mixture of $He^3$ and $He^4$ liquids (1)

(ii) Theory of cooling

Below triple point $T = 0.86\,K$ $He^3$ and $He^4$ liquids form a 2 phase system (1) — $He^3$ rich phase
$He^4$ rich phase

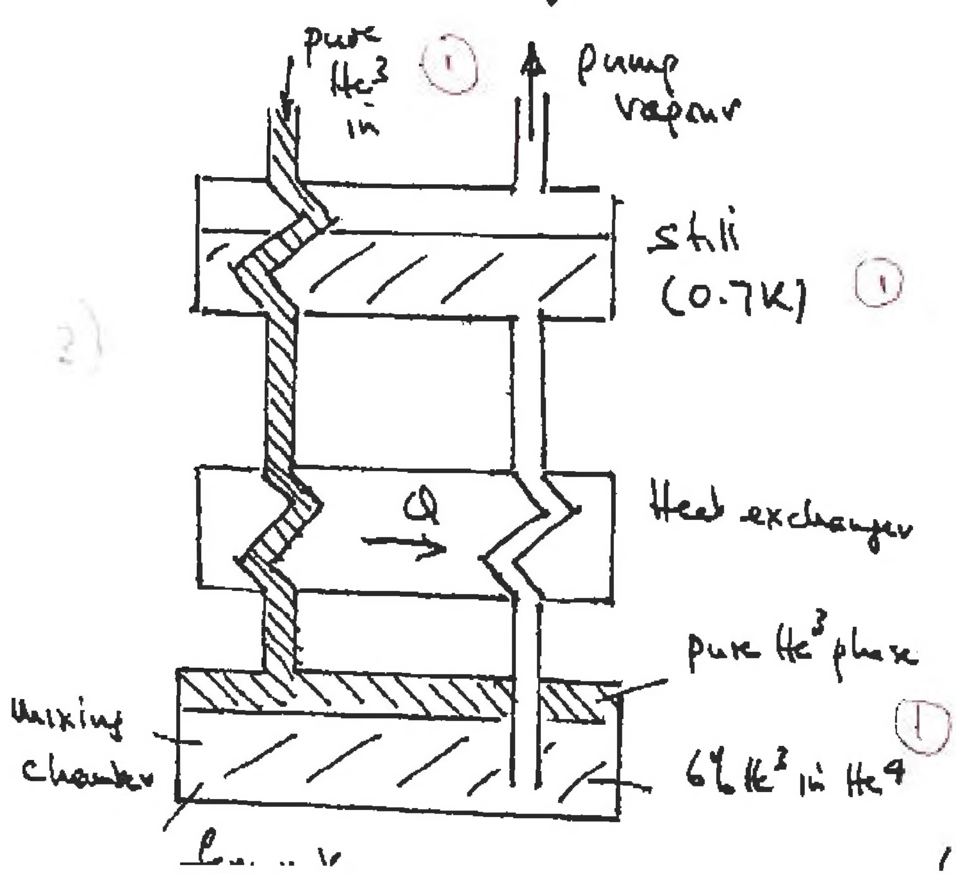
At $T=0$ $He^3$ rich phase $\rightarrow$ 100% $He^3$ (1)
$He^4$ rich phase $\rightarrow$ 6% $He^3$ + 94% $He^4$ (1)

Cooling.

In mixing chamber (see diagram) phase of $He^3$ floats on top of $He^4$ (6% $He^3$) phase.

Cooling occurs at phase boundary as $He^3$ atoms evaporate from $He^3$ rich to $He^4$ rich phases. This takes energy from $He^3$ rich phase and cools it (1)

(iii) Schematic diagram




(iv) Experimental process.

Apparatus works by allowing above evaporation to occur continuously. (in cycles)

Dilute phase pumped – (1) pure $He^3$ gas comes off and is circulated down thru' heat exchanger to pure $He^3$ layer in mixing chamber – continuous process.

(1) (v) Starting ...

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